Searching and Hashing
Sequential Search

Property: Sequential search (array implementation) uses \( N+1 \) comparisons for an unsuccessful search (always).

Unsuccessful Search:
\[ \Theta(n) \]

Successful Search: item is in any location of the array
\[ O(n) \]
\[ \Omega(1) \]

Property: Sequential search (array implementation) uses about \( n/2 \) comparisons for a successful search (on the average).

\[ \sum_{i=1}^{n} i = \frac{(n^2 + n)}{2} \]

Average-Case: The number of key comparisons 1,2,...,n
\[ O(n) \]
Binary Search – Analysis

• **Property:** Binary search never uses more than $\log_2 N + 1$ comparisons for either successful or unsuccessful search.

• For an unsuccessful search:
  – The number of iterations in the loop is $\lfloor \log_2 n \rfloor + 1$
    $$\Rightarrow \Theta(\log_2 n)$$

• For a successful search:
  – The number of iterations is $\lfloor \log_2 n \rfloor + 1$
    $$\Rightarrow O(\log_2 n)$$
    $$\Rightarrow \Omega(1)$$

– **Average-Case:** The avg. # of iterations < $\log_2 n$ $$\Rightarrow O(\log_2 n)$$

0 1 2 3 4 5 6 7 \(\leftarrow\) an array with size 8
3 2 3 1 3 2 3 4 \(\leftarrow\) # of iterations

The average # of iterations = $21/8 < \log_2 8$

$T(n) = T(n/2) + 1$ with $T(1) = 1$

$$\Rightarrow O(\log_2 n)$$
Interpolation Search

- One improvement possible in binary search is to try to guess more precisely where the key being sought falls within the current interval of interest (rather than blindly using the middle element at each step), i.e., looking up a number in telephone directory.

- Note that $x = (l+r) \text{ div } 2$ is derived from the expression
  \[ x = l + \frac{1}{2} (r - l) \]

- In the interpolation search
  \[ x = l + \left(\frac{v - a[l].key}{a[r].key - a[l].key}\right) * (r-l) \]
  might be a better guess.

- This assumes numerical evenly distributed key values.
Interpolation Search

Example:

A A A C E E E G H I L M N P R S X

- If look for $v = M$, the first table position examined would be 9, since

$$x = l + \left(\frac{v - a[l].key}{a[r].key - a[l].key}\right) \times (r-l)$$

$$x = 1 + \left(\frac{13 - 1}{24 - 1}\right) \times (17-1) = 9.3...$$

I L M N P R S X

M N P R S X

Property: Interpolation search uses fewer than $\log\log n + 1$ comparisons for both successful and unsuccessful search, in files of random keys. $\rightarrow O(\log\log n)$

$$\log\log N < 5 \text{ if } N = 10^9.$$ 

Suggestion: Interpolation search certainly should be considered for large files, for applications where comparisons are particularly expensive, or for external methods where very high access costs are involved.
Hashing

- A **bucket** is a unit of storage containing one or more records (a bucket is typically a disk block).
- The file blocks are divided into **M equal-sized buckets**, numbered bucket$_0$, bucket$_1$, ..., bucket$_{M-1}$. Typically, a bucket corresponds to one (or a fixed number of) disk block.
- In a **hash file organization** we obtain the bucket of a record directly from its search-key value using a **hash function**, h(K).
- The record with hash key value K is stored in bucket$_i$, where i=h(K).
- **Hash function** is used to locate records for access, insertion as well as deletion.
- Records with different search-key values may be mapped to the same bucket; thus entire bucket has to be searched sequentially to locate a record.
Static Hashing

- # primary pages fixed, allocated sequentially, never de-allocated; overflow pages if needed.
- $h(K) \mod M = \text{bucket to which data entry with key } k \text{ belongs.} \ (M = \# \of \text{buckets})$

![Diagram of static hashing]

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Hash functions

Solution: Use a hash function $h$ to map the universe $U$ of all keys into $\{0, 1, \ldots, m-1\}$:

When a record to be inserted maps to an already occupied slot in $T$, a collision occurs.
Static Hashing

• One of the file fields is designated to be the hash key, \( K \), of the file.

• **Collisions** occur when a new record hashes to a bucket that is already full.

• An **overflow file** is kept for storing such records. Overflow records that hash to each bucket can be linked together.

• To **reduce overflow records**, a hash file is typically kept 70-80% full.

• The hash function \( h \) should distribute the records uniformly among the buckets; otherwise, search time will be increased because many overflow records will exist.
Handling of Bucket Overflows (Cont.)

- **Overflow chaining** – the overflow buckets of a given bucket are chained together in a linked list.

- An alternative, called **open hashing**, which does not use overflow buckets, is not suitable for database applications.
Resolving collisions by chaining

- Records in the same slot are linked into a list.

\[ h(49) = h(86) = h(52) = i \]
Analysis of chaining

We make the assumption of *simple uniform hashing*:
- Each key \( k \in K \) of keys is equally likely to be hashed to any slot of table \( T \), independent of where other keys are hashed.

Let \( n \) be the number of keys in the table, and let \( m \) be the number of slots.

Define the *load factor* of \( T \) to be
\[
\alpha = \frac{n}{m}
\]
= average number of keys per slot.
Search cost

Expected time to search for a record with a given key $= \Theta(1 + \alpha)$.

- apply hash function and access slot
- search the list

Expected search time $= \Theta(1)$ if $\alpha = O(1)$, or equivalently, if $n = O(m)$. 
Resolving collisions by open addressing

No storage is used outside of the hash table itself.
- Insertion systematically probes the table until an empty slot is found.
- The hash function depends on both the key and probe number:
  \[ h : U \times \{0, 1, \ldots, m-1\} \rightarrow \{0, 1, \ldots, m-1\}. \]
- The probe sequence \( \langle h(k,0), h(k,1), \ldots, h(k,m-1) \rangle \) should be a permutation of \( \{0, 1, \ldots, m-1\} \).
- The table may fill up, and deletion is difficult (but not impossible).
Example of open addressing

Insert key $k = 496$:

0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$
Example of open addressing

Search for key $k = 496$:

0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$

Search uses the same probe sequence, terminating successfully if it finds the key and unsuccessfully if it encounters an empty slot.
Probing strategies

**Linear probing:**

Given an ordinary hash function $h'(k)$, linear probing uses the hash function

$$h(k,i) = (h'(k) + i) \mod m.$$ 

This method, though simple, suffers from *primary clustering*, where long runs of occupied slots build up, increasing the average search time. Moreover, the long runs of occupied slots tend to get longer.
Linear Probing: Example

\[ h(k, i) = (h'(k) + i) \mod M \]

**Ex:** \( M = 10 \)

**Input:** \(<5, 3, 6, 13, 23, 15>\)

\[ h(3, 0) = (h'(3) + 0) \mod 10 = 3 \]
\[ h(13, 0) = (h'(13) + 0) \mod 10 = 3 \]
\[ h(13, 1) = (h'(13) + 1) \mod 10 = 4 \]
\[ h(5, 0) = (h'(5) + 0) \mod 10 = 5 \]
\[ h(6, 0) = (h'(6) + 0) \mod 10 = 6 \]
\[ h(23, 0) = (h'(23) + 0) \mod 10 = 3 \]
\[ h(23, 1) = (h'(23) + 1) \mod 10 = 4 \]

---

\[ h(23, 4) = (h'(23) + 4) \mod 10 = 7 \]
Quadratic Probing

- Uses a hash function of the form

\[ h(k, i) = (h'(k) + c_1 i + c_2 i^2) \mod M \]

Where \( h' \) is an auxiliary hash function, \( c_1 \) and \( c_2 \neq 0 \) are auxiliary constants and \( i = 0, 1, ..., M-1 \).

- Initial position probed is \( T[h'(k)] \).

- If two keys have the same initial probe position, then their probe sequences are the same, since \( h(k_1,0) = h(k_2,0) \)’ implies \( h(k_1,i) = h(k_2,i) \)’. This leads to a milder form of clustering, secondary clustering.
Quadratic Probing: Example

\[ h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod M \]

**Ex:** \( M = 10, \ c_1 = 2, \ c_2 = 1 \)

Input = \(<2, 3, 22, 12, 18>\)

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\[ h(2,0) = (h'(2) + 2*0+1*0) \mod 10 = 2 \]
\[ h(3,0) = (h'(3) + 2*0+1*0) \mod 10 = 3 \]
\[ h(22,0) = (h'(22)+2*0+1*0) \mod 10 = 2? \]
\[ h(22,1) = (h'(22)+2*1+1*1) \mod 10 = 5 \]
\[ h(12,0) = (h'(12)+2*0+1*0) \mod 10 = 2? \]
\[ h(12,1) = (h'(12)+2*1+1*1) \mod 10 = 5? \]
\[ h(12,2) = (h'(12)+2*2+1*4) \mod 10 = 0 \]
\[ h(18,0) = (h'(18) + 2*0+1*0) \mod 10 = 8 \]
Probing strategies

Double hashing

Given two ordinary hash functions $h_1(k)$ and $h_2(k)$, double hashing uses the hash function

$$h(k,i) = (h_1(k) + i \cdot h_2(k)) \mod m.$$ 

This method generally produces excellent results, but $h_2(k)$ must be relatively prime to $m$. One way is to make $m$ a power of 2 and design $h_2(k)$ to produce only odd numbers.

Double hashing represents an improvement, since each $(h_1(k), h_2(k))$ pair yields a distinct probe sequence.
Double Hashing

\[ h(k,i) = (h_1(k) + i h_2(k)) \mod M \]

Either \( M = 2^d \) and design \( h_2 \) so that it produces odd numbers or \( M \) is prime and \( h_2 \) produces positive integer less than \( M \).

Ex: \( M = 13 \), \( h_1(k) = k \mod M \), \( h_2(k) = 1 + (k \mod M') \).

Input = \( <96, 79, 14> \)

\[ h_1(96,0) = 98 \mod 13 = 5 \]
\[ h_1(79,0) = 79 \mod 13 = 1 \]
\[ h_1(14,0) = 14 \mod 13 = 1 \]?

\[ h_2(14) = 1 + 14 \mod 11 = 4 \]
\[ h(14, 1) = (h_1(14) + i \times h_2(14)) \mod 13 = 1+1.4=5 \]?
\[ h(14, 2) = (1 + 2\times 4) \mod 13 = 9 \]
Analysis of open addressing

We make the assumption of uniform hashing:

- Each key is equally likely to have any one of the $m!$ permutations as its probe sequence.

**Theorem.** Given an open-addressed hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.
Proof of the theorem

Proof.
• At least one probe is always necessary.
• With probability $\frac{n}{m}$, the first probe hits an occupied slot, and a second probe is necessary.
• With probability $\frac{(n-1)}{(m-1)}$, the second probe hits an occupied slot, and a third probe is necessary.
• With probability $\frac{(n-2)}{(m-2)}$, the third probe hits an occupied slot, etc.

Observe that $\frac{n-i}{m-i} < \frac{n}{m} = \alpha$ for $i = 1, 2, \ldots, n$. 
Proof (continued)

Therefore, the expected number of probes is

\[ 1 + \frac{n}{m} \left( 1 + \frac{n-1}{m-1} \left( 1 + \frac{n-2}{m-2} \left( \ldots \left( 1 + \frac{1}{m-n+1} \right) \ldots \right) \right) \right), \]

\[ \leq 1 + \alpha (1 + \alpha (1 + \alpha (\ldots (1 + \alpha )\ldots ))) \]

\[ \leq 1 + \alpha + \alpha^2 + \alpha^3 + \ldots \]

\[ = \sum_{i=0}^{\infty} \alpha^i \]

\[ = \frac{1}{1-\alpha}. \]

\textit{The textbook has a more rigorous proof.}

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Implications of the theorem

• If $\alpha$ is constant, then accessing an open-addressed hash table takes constant time.
• If the table is half full, then the expected number of probes is $\frac{1}{1-0.5} = 2$.
• If the table is 90% full, then the expected number of probes is $\frac{1}{1-0.9} = 10$. 
Dynamic Hashing Techniques

- Hashing techniques are adapted to allow the \textit{dynamic growth} and \textit{shrinking} of the number of file records.
- These techniques include the following: \textit{dynamic hashing}, \textit{extendible hashing}, and \textit{linear hashing}.
- These hashing techniques use the binary representation of the hash value $h(K)$.
- In \textit{dynamic hashing} the directory is a binary tree.
- In \textit{extendible hashing} the directory is an array of size $2^d$ where $d$ is called the \textit{global depth}. 
Linear Hashing

- This is another dynamic hashing scheme, an alternative to Extendible Hashing.
- LH handles the problem of long overflow chains without using a directory, and handles duplicates.

**Idea:** Use a family of hash functions $h_0, h_1, h_2, ...$

- $h_i(key) = h(key) \mod (2^iN)$; $N =$ initial # buckets
- $h$ is some hash function
- If $N = 2^{d0}$, for some $d0$, $h_i$ consists of applying $h$ and looking at the last $di$ bits, where $di = d0 + i$.
- $h_{i+1}$ doubles the range of $h_i$ (similar to directory doubling)
Overview of LH File

- In the middle of a round.

Buckets that existed at the beginning of this round: this is the range of \( h_{\text{Level}} \)

Bucket to be split

Next

Buckets split in this round: if \( h_{\text{Level}} \) (search key value) is in this range, must use \( h_{\text{Level}+1} \) (search key value) to decide if entry is in `split image' bucket.

`split image' buckets: created (through splitting of other buckets) in this round
A sample problem in LH

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67: 100 0011  
43: 010 1011  
79: 100 1111  
15: 000 1111  
19: 001 0011  
27: 001 1011  
64: 100 0000  
12: 000 1100  
33: 010 0001  
57: 011 1001  
65: 100 0001
Example: LH

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LH

- **Linear Hashing** avoids directory by splitting buckets round-robin, and using overflow pages.
  - Overflow pages not likely to be long.
  - Duplicates handled easily.
  - **Space utilization** could be lower than Extendible Hashing, since splits not concentrated on `dense` data areas.

- For hash-based indexes, a *skewed* data distribution is one in which the *hash values* of data entries are not uniformly distributed!
Hash function returns $b$ bits
Only the prefix $i$ bits are used to hash the item
There are $2^i$ entries in the bucket address table
Let $i_j$ be the length of the common hash prefix for data bucket $j$, there is $2^{(i-i_j)}$ entries in bucket address table points to $j$.
In this structure, $i_2 = i_3 = i$, $2^0 = 1$ entry, whereas $i_1 = i - 1$, $2^1 = 2$ entries.
Example: Extendable Hashing

- Example 5: Suppose the hash function is $h(x) = x \mod 8$ and each bucket can hold at most two records. Show the extendable hash structure after inserting 1, 4, 5, 7, 8, 2, 20.
Example: Extendable Hashing

inserting 1, 4, 5, 7, 8, 2, 20
Suppose the hash function \( h(x) = x \mod 8 \), each bucket can hold at most 2 records. Show the structure after inserting “20”
Example: Extendable Hashing